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LETTER TO THE EDITOR

Series expansion studies of directed percolation: estimates of the correlation length exponents

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Abstract. Series expansion methods have been used to estimate the longitudinal and transverse correlation length exponents for directed (bond) percolation on the square lattice. The central estimate for the ratio of these exponents $\phi = 1.579 \pm 0.006$ is in good agreement with the value obtained by the phenomenological scaling renormalisation group method.

The directed percolation problem first introduced by Broadbent and Hammersley (1957) has recently proved to be of considerable interest. Blease (1977a, b, c) asked whether the directed and undirected percolation problems were members of the same universality class. His results from series expansions, and more recent results from Monte Carlo simulations (Kertesz and Vicsek 1980, Dhar and Barma 1981), for various lattices, indicate that models in which percolation takes place in a 'preferred' flow direction are members of a new universality class. On the basis of a diagrammatic expansion, Obukhov (1980) argued that $d = 5$ was the upper critical dimension for such models and derived the critical exponents to leading order in $\varepsilon = 5 - d$. In addition, Cardy and Sugar (1980) have shown that there is an exact mapping between the directed percolation problem and Reggeon field theory which models the creation, propagation and destruction of a cascade of elementary particles.

Kinzel and Yeomans (1981) have pointed out that two correlation lengths $\xi_\parallel$ and $\xi_\perp$ must be considered in directed percolation. $\xi_\parallel$ and $\xi_\perp$ typify the decay of the pair connectedness parallel to and perpendicular to the flow, and have critical exponents $\nu_\parallel$ and $\nu_\perp$ respectively. Kinzel and Yeomans (1981) were able to obtain accurate estimates of $\nu_\parallel$ and $\nu_\perp$, and hence the anisotropy parameter $\phi = \nu_\parallel/\nu_\perp$, by using the phenomenological renormalisation group approach of Nightingale (1976), for the square lattice in which all parallel bonds are directed in the same way.

Blease (1977c) estimated $\nu_\parallel$ for this lattice (also the corresponding triangular and honeycomb lattices) from an analysis of the zeroth and second moments of the pair connectedness $C(0, j)$ (the probability that site j is connected to the origin):

$$\mu_0 = \sum_j C(0, j) \underset{p \to p_c}{\sim} (p_c - p)^{-\gamma}, \tag{1a}$$

$$\mu_2^{(r)} = \sum_j C(0, j) r_j^2 \underset{p \to p_c}{\sim} (p_c - p)^{-\gamma - 2\nu_\parallel}, \tag{1b}$$

where γ is mean size exponent and $\boldsymbol{r}_j$ the position vector of site j. There is however no direct estimate of $\nu_\perp$ for directed percolation available apart from that of Kinzel and Yeomans (1981) (estimates have been obtained for lattice models thought to have the same critical behaviour as Reggeon field theory: Brower *et al* (1978), Grassberger and de la Torre (1979)).

Writing $\boldsymbol{r}_j \equiv (t_j, y_j)$, where the t and y axes are parallel to and perpendicular to the direction of flow respectively, the moments $\mu_2^{(t)}$ and $\mu_2^{(y)}$ may be defined as

$$\mu_2^{(t)} = \sum_j C(0, j) t_j^2 \underset{p \to p_c}{\sim} (p_c - p)^{-\gamma - 2\nu_\parallel}, \tag{2a}$$

$$\mu_2^{(y)} = \sum_j C(0, j) y_j^2 \underset{p \to p_c}{\sim} (p_c - p)^{-\gamma - 2\nu_\perp}. \tag{2b}$$

Using a similar computational technique to that described by Blease (1977c), the coefficients of the terms in the low-density expansions of $\mu_2^{(y)}$ and $\mu_2^{(t)}$, up to $O(p^{22})$, were derived for the bond percolation problem on the square lattice considered by Kinzel and Yeomans (1981), and are presented in table 1. (The series for μ_0 has been extended by one term and is also presented in table 1 for completeness.)

Blease (1977c) concluded that the critical probability was $p_c = 0.6446 \pm 0.0002$ using the mean size series. We have used this result to form biased estimates of $\nu_\parallel$ and $\nu_\perp$ using Dlog Padé approximants to the series for the ratios $\mu_2^{(t)}/\mu_0$ and $\mu_2^{(y)}/\mu_0$ obtained from the series in table 1. By inspecting the Padé tables for the assumed values 0.6444,

Table 1. Coefficients of p^m in the low-density expansions of the pair connectedness moments.

m	$\mu_0^\dagger$	$\mu_2^{(y)}$	$\mu_2^{(t)}$
0	1	0	0
1	2	1	1
2	4	4	8
3	8	12	36
4	15	32	126
5	28	78	382
6	50	179	1 047
7	90	393	2 681
8	156	832	6 484
9	274	1 717	15 069
10	466	3 451	33 723
11	804	6 828	73 524
12	1 348	13 232	155 970
13	2 300	25 386	324 930
14	3 804	47 877	662 617
15	6 450	89 721	1 334 065
16	10 547	165 647	2 639 033
17	17 784	304 748	5 173 100
18	28 826	553 053	9 988 505
19	48 464	1 002 426	19 164 778
20	77 689	1 793 437	36 273 493
21	130 868	3 211 514	68 392 722
22	207 308	5 675 637	127 298 209

† This series was given up to $m = 21$ by Blease (1977c).

0.6446 and 0.6448 of p_c it was found that

$$\nu_\parallel = 1.733 + 45\Delta p_c \pm 0.004 \qquad \text{and} \qquad \nu_\perp = 1.0978 + 25\Delta p_c \pm 0.0008$$

where Δp_c is the shift from the assumed central value 0.6446. Hence

$$\phi = 1.579 + 5\Delta p_c \pm 0.005.$$

Using $|\Delta p_c| \leq 0.0002$, we conclude that

$$\nu_\parallel = 1.733 \pm 0.013, \qquad \nu_\perp = 1.098 \pm 0.006, \qquad \phi = 1.579 \pm 0.006.$$

This is in good agreement with the value of

$$\phi = 1.582 \pm 0.001$$

obtained by Kinzel and Yeomans (1981).

In summary, estimates of $\nu_\parallel$, $\nu_\perp$ and hence the anisotropy parameter ϕ may be obtained from the low-density series expansions of the moments defined in equations (2). The results obtained in this way for bond percolation on the directed square lattice are in good agreement with those obtained by phenomenological rescaling (Kinzel and Yeomans 1981). Work is in progress to extend these results to other lattices.

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